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Tables for Boundary-Layer Thicknesses of Similar Compressible Laminar Flow

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Similarity solution of the compressible, laminar boundary-layer equation depends on pressure gradient parameter β and wall to inviscid stagnation temperature ratio g_w. However, the derived quantities, such as various thicknesses, also depend on speed parameter S, thereby requiring three dimensional tables for the tabulated results. A new formulation is provided that enables all quantities of interest to be determined by the two-dimensional tables in which β and g_w are the input parameters. With such a set, accurate values can be found for the skin-friction coefficient, Stanton number, and the five most common viscous and thermal boundary-layer thicknesses for arbitrary values of the speed parameter. A comprehensive set of tables is provided in which β ranges from its separation value to 100 and g_w ranges 0 to 5. Quasi-linearization method is applied to the governing equations and generalized Newton-Raphson method is used to obtain successive initial condition. As a result computation time is reduced significantly.

Key Words: Laminar Flow, Boundary Layer Thickness, Compressible Flow, Similarity Solution

1. Introduction

Major trend of recent research in fluid mechanics is a direct numerical simulation of Euler or Navier-Stokes equation. In high speed flow or in violent situation the flow becomes turbulent and the research on turbulent flow have made significant successes. Due to the fast advance in growth of computing power, a direct numerical simulation of turbulent Navier-Stokes equation is available. However it is still a time-consuming costly job to run the code of the 2-D or 3-D Navier-Stokes equations. Also many flow problems sit within a laminar flow range. In this regard, it is worthwhile to make the lamainr solution of the Navier-Stokes equations easy to use. The purpose of this paper is to provide a method of obtaining the boundary layer properties without resorting to running a computer. The final result will be several tables which are used to calculate the boundary layer properties by algebraic relations.

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The similarity solutions of the first-order, laminar boundary-layer equation have been of great interest in fluid mechanics since the early work of Blasius. This paper is concerned with the compressible similarity equation under the traditional assumptions and such that the flow is a steady two dimensional or axisymmetric of perfect gas with unity for the Prandtl number and the Chapman-Rubesin parameter, and the bounding wall is impermeable.

There are innumerable publications dealing with one or two aspects of this theory, including the occasional presentation of tabulated results. These come in two forms: (a) parameters directly involved in the solution of the differential equations, such as the skin-friction parameter at the wall, and (b) the derived parameter, such as a momentum thickness. Most of the derived parameters depend on a speed parameter S. defined by

Eq. (A-3), as well as the pressure gradient parameter β, and a stagnation enthalpy ratio g_w. A comprehensive table of the results is therefore three dimensional and, in fact, doesn't exist. To our knowledge, one of the more extensive tabulations is due to Back (1970). His three-dimensional table, however, is scanty lacking results for negative β and for $g_w > 1$. Furthermore, the spacing on g_w and S is nonuniform and inadequate for interpolation or extrapolation. These remarks are not criticisms, since a comprehensive three-dimensional table is a prohibitive undertaking, and even if it existed, would be difficult and awkward to use.

In Sec. 2 and in Appendix A, the definitions and equations corresponding to the assumptions in the opening paragraph are formulated. Since this material (Back, 1970; Libby and Liu, 1968; Pade et al., 1985; Dewey and Gross, 1967) is fairly standard, the presentation is succinct. In Sec. 3, we will present the analysis described in Ref. (Bae and Emanuel, 1989) briefly, which shows that a three-dimensional table is unnecessary by using a derivation reminiscent of that used to obtain the boundary-layer integral equations. Results for the usual boundary-layer parameters of interest, including the skin-friction coefficient, Stanton number, and five thicknesses, can be comprehensively tabulated using only two-dimensional tables. We provide this set of tables that they hold for arbitrary values of S and for

$$\beta_{sp} \leq \beta \leq 100, \quad 0 \leq g_w \leq 5 \tag{1}$$

where β_{sp} is the value of β that first corresponds to a zero wall skin-friction (or separation) value. Tables that provide a wide range of β and g_w values are occasionally useful. For instance, in a cryogenic flow large g_w values may be encountered, while large β values (Dewey and Gross, 1967; Wortman, 1987) can occur in nozzle throat with a small radius of curvature or on the shoulder of a blunt reentry vehicle, The final section discusses special cases and our results.

2. Formulation

The compressible boundary-layer equations

$$f''' + ff'' + \beta[g_w + (1-g_w)G - f'^2] = 0 \tag{2a}$$

$$G'' + fG' = 0 \tag{2b}$$

are subject to the boundary conditions

$$f(0) = f'(0) = 0, \; f'(\infty) = 1 \tag{2c}$$

$$G(0) = 0, \; G(\infty) = 1 \tag{2d}$$

A numerical solution to Eqs. (2) requires the prescribed values for β and g_w. In this paper, these values are constrained by the requirement that $f''_w \geq 0$ and by relations (1), where $f''_w \geq 0$ is zero when $\beta = \beta_{sp}$. For the β values chosen, the solution is unique.

3. Analysis

In addition to the transformed edge values η_{ev} and η_{et} Eqs. (A-7) and (A-8) show that the Stanton number St, and the skin-friction coefficient c_f depend only on and g_w. Our objective is to determine formulas for $\gamma\psi$, such that the dependence on S is analytically explicit. Here, ψ is any one of the boundary-layer thicknesses defined by Eqs. (A-9) through (A-13), while γ is defined by Eq. (A-6). Thus, only tables where β and g_w are the entree values would be needed.

For this objective, two parameters are defined.

$$C_v(\beta, g_w) = \int_0^\infty (1-f')\, d\eta = \lim_{\eta\to\infty} (\eta - f) \tag{3}$$

$$C_t(\beta, g_w) = \int_0^\infty (1-G)\, d\eta = \lim_{\eta\to\infty} \left(\eta - \int_0^\infty G d\eta\right) \tag{4}$$

where C_v is widely used in the incompressible theory. A number of integrals are also needed.

The infinite upper limit is replaced in Eqs. (A-11)-(A-13) with n and numerically integrated. Whenever a G integral or an f appears, these are replaced by Eqs. (3) and (4). After simplification, the $\eta \to \infty$ limit is then taken. The tedious mathematical manipulation will not be reiterated here.

As an example, only the procedure for the displacement thickness is presented. The displacement thickness, Eq. (A-11) becomes

$$\gamma\delta^* = \lim_{\eta\to\infty} \int_0^\infty \left[\frac{g_w + (1-g_w)G - Sf'^2}{1-S} - f'\right] d\eta$$

where S and γ are defined in (A-3) and (A-6).

Equation (B-2) is used for the Sf'^2 term, while Eq. (3b) is used for the two G integrals, with the result

$$\gamma\delta^* = \lim_{\eta\to\infty}\left\{\frac{1}{1-S}\left[g_w\eta + (1-g_w)(\eta - C_t) - \frac{S}{(1+\beta)}(f'' + ff' + \beta g_w \eta - f''_w) - \frac{S\beta(1-g_w)}{1+\beta}(\eta - C_t)\right] - f\right\}$$

Replace f with Eq. (3a) and set

$$f'(\infty)=1,\quad f''(\infty)=0$$

to obtain

$$\gamma\delta^* = \lim_{\eta\to\infty}\left(\frac{1}{1-S}\left(\eta - (1-g_w)C_t - \frac{S}{1+\beta}[\eta - C_v + \beta g_w \eta - f''_w + \beta(1-g_w)\eta - \beta(1-g_w)C_t]\right\} + C_v - \eta\right)$$

On the right side, the η term cancel, leaving

$$\gamma\delta^* = \frac{1}{(1+\beta)(1-S)}\cdot \{Sf''_w + [1+(1-S)\beta][C_v - (1-g_w)C_t]\} \qquad (5)$$

where the relations given in Appendix B are utilized.

In a similar manner, the remaining four boundary-layer thicknesses (A-9), (A-10), (A-12), and (A-13) are transformed such as

$$\gamma\delta = \eta_{ev} + \frac{1}{(1+\beta)(1-S)}\{S(f''_w + C_v) - (1-g_w)[1+\beta(1-S)]C_t]\} \qquad (6)$$

$$\gamma\delta_t = \eta_{et} + \frac{1}{(1+\beta)(1-S)}\{S(f''_w + C_v) - (1-g_w)[1+\beta(1-S)]C_t]\} \qquad (7)$$

$$\gamma\theta = \frac{1}{1+\beta}\{f''_w - \beta[C_v - (1-g_w)C_t]\} \qquad (8)$$

$$\gamma\phi = G'_w \qquad (9)$$

for θ and ϕ, δ and δ_t. By tabulating η_{ev}, η_{et}, C_v, C_t, f''_w, and G'_w in terms of β and g_w, the foregoing thicknesses can be found for an arbitrary value of S. In addition, it is convenient to also tabulate β_{sp} (and 0.5 β_{sp}) versus g_w. Tables 1 through 7 provide these results.

Table 1 β_{sp} vs g_w

g_w	β_{sp}	$0.5\beta_{sp}$	β_{sp}[+]
0.0	-0.32650	-0.16325	-0.326
0.2	-0.30865	-0.15433	-0.3088
0.4	-0.27783	-0.13892	--
0.6	-0.24757	-0.12379	-0.246
0.8	-0.22115	-0.11058	--
1.0	-0.19884	-0.09942	-0.1988
1.5	-0.15735	-0.07867	--
2.0	-0.12950	-0.06475	-0.1295
3.0	-0.09521	-0.04760	--
4.0	-0.07511	-0.03756	--
5.0	-0.06199	-0.03099	--

[+]See Ref. in Mcleod and Serrin (1968)

An overshoot is experienced by f' whenever $\beta > 0$ and $g_w > 1$. Figure 1 shows this behavior for various g_w when $\beta = 1.5$. There is a considerable velocity overshoot for large g_w, while for a gw= 1.5 the overshoot is small. When there is no overshoot, or the overshoot is too small to be discernible, we use a conventional definition for η_{ev}

$$f'(\eta_{ev}) = 0.99,\ f'(\eta_m) \le 1.000001 \qquad (10a)$$

where η_m is the η value where f' is a maximum. For a discernible maximum, the smaller of the two η_{ev} given by

$$f'(\eta_{ev}) = =0.9 + 0.1f'(\eta_m),\ \eta_{ev} > \eta_m \qquad (10b)$$

$$f'(\eta_{ev}) = 1.01,\ \eta_{ev} > \eta_m \qquad (10c)$$

is used. To the left of Shaded region in Table 4, Eq. (10a) defines η_{ev}. To the right of Shaded region, Eq. (10c) holds, whereas in the Shaded region Eq. (10b) is used.

4. Numerical Method

Equations (2a) and (2b) are reduced to five, first-order differential equations that are integrated by means of a fourth-order Runge-Kutta method using a $\Delta\eta = 0.01$ step size. The computations were performed on an IBM 3081 in double precision. For $\beta \le 1$, computation times, per case, ranged from 15 to 25 sec. For a β of 100, this time

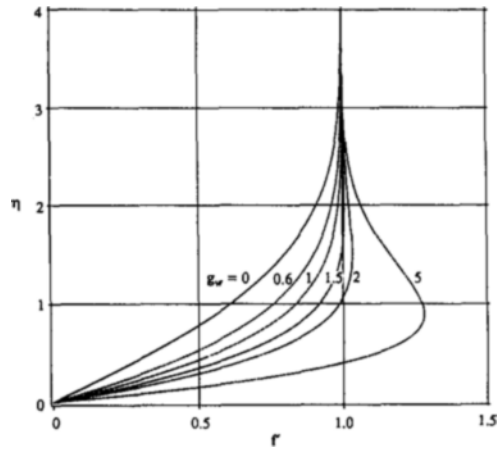

Fig. 1 Velocity profile vs η for $\beta=1.5$.

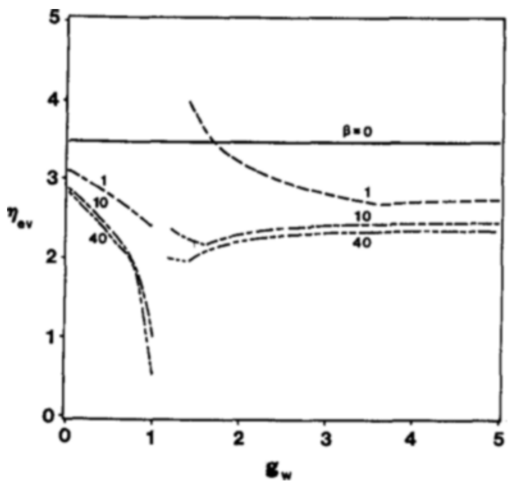

Fig. 2 Velocity thickness η_{ev} vs g_w.

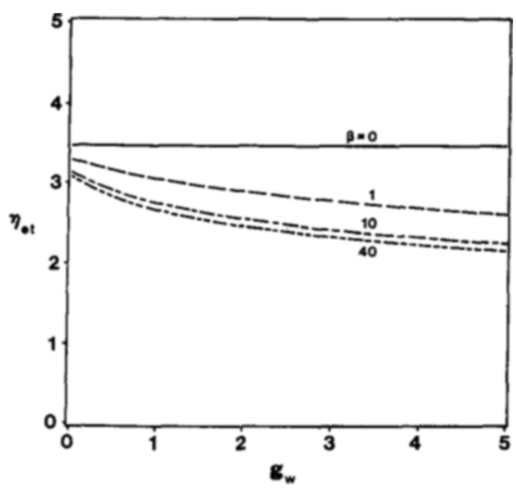

Fig. 3 Thermal thickness η_{et} vs g_w.

increased to about 180 sec.

Quasilinearization (Libby and Liu, 1968; Radbill, 1964; Libby and Chen, 1966) is used; however, our method most closely resembles that of Back. (1970) The equations are first integrated to a prescribed value of η, denoted as η_a, with guessed initial values. The η_a parameter ranges from 6, when $\beta>20$, to 9, when $\beta\leq 1$. The initial values are successively corrected by a generalized Newton-Raphson method (McGill and Kenneth, 1964) until the infinity conditions, $f'(\eta_b)=1$ and $G(\eta_b)=1$, $\eta_b\geq\eta_a$ are satisfied. The convergence criteria are

$$|1-f'_{k+1}(\eta_b)|<\varepsilon_1,\quad |1-G'_{k+1}(\eta_b)|<\varepsilon_1 \tag{11a}$$

$$|f''_{w,k+1}-f''_{w,k}|<\varepsilon_2,\quad |G''_{w,k+1}-G''_{w,k}|<\varepsilon_2 \tag{11b}$$

where k is the iteration index, and $\varepsilon_1=10^{-5}$, $\varepsilon_2=10^{-7}$.

when $g_w\leq 1$, or when $0\leq\beta\leq 20$ for all g_w, the correlation formulas (Pade, Postan, Anshelovitz and Wolfstein, 1985)

$$f''_w=(4g_w\beta/3)^{1/2} \tag{12}$$

$$g'_w=0.4696+0.2g_w^{0.09}(1-g_w^{1.8})\exp(-\beta^{-0.3}) \tag{13}$$

are employed for the initial guesses. Elsewhere, the initial conditions of the nearest β or g_w solution are used.

For $\beta>1$, convergence is difficult because $f'(\eta_b)$ is sensitive to f''_w.

A band

$$1-A\leq f'(\eta_b)\leq 1+B \tag{14}$$

is set, where $A=B=0.5$ when $g_w\leq 1$, when $g_w>1$, B is increased to accommodate the velocity overshoot. The value for f''_w is corrected, with G'_w fixed, until $f'(\eta_a)$ falls within the band. Once this occurs, the Newton-Raphson method is used to modify both f''_w and G'_w. Because of the accelerated rate of convergence, due to the Newton-Raphson procedure, usually five, or fewer, iterations are sufficient, not counting intermediate f''_w corrections with G'_w fixed.

5. Discussion

Figures 2 and 3 show η_{ev} and η_{et}, respectively, for several β values. As it is evident, the two edge values dramatically differ in magnitude and trend. As a consequence, the approximations used in obtaining Eqs. (6) and (7) can be questioned. These approximations involved replacing $\eta \to \infty$ with $\eta=\eta^*$ in Eqs. (3) and evaluating f' and f'' at η^* instead of infinity. The accuracy of δ and δ_t is established by evaluating the relative errors.

$$E_v=\frac{|\delta_{ex}-\delta|}{\delta_{ex}}\times 10^2,\ E_t=\frac{|\delta_{tex}-\delta_t|}{\delta_{tex}}\times 10^2$$

for each of the 231 cases in Tables 2~7 at S values of 0, 0.5, and 0.9.All 693 Ev values are below 0.9%. The largest values for E_t occur when $S=0.9$ and $\beta=\beta_{sp}$. These are 1.40% and 1.11% when $g_w=0$ and 0.2, respectively. All other E_t values are below 0.9%, usually considerably so.

An important reason for the uniformly small E_t values is Eq. (10b). Other relations were tried, including a smooth interpolation, but resulted in substantially larger E_V values. With Eq. (10b), the maximum E_t value in the middle region, between the dashed lines, of Table 4 is only 0.285%. As a consequence of Eq. (10b), however, η_{ev} does not have a smooth variation when $g_w>1$, as shown in Fig. 2. As evident from Fig. 1, when $g_w\leq 1$, η_{ev} decreases smoothly with increasing g_w for a fixed β value. The transition from Eq. (10a) to Eq. (10b), which occurs near $g_w=1$, is generally discontinuous. A second transition in Fig. 2 occurs at a larger g_w value in which the slope is discontinuous. This discontinuity stems from the transition from Eq. (10b) to (10c). Only η_{ev} is subject to this type of behavior; all the parameters in the other tables have smooth variations.

Another assessment of the accuracy is obtainable from the β_{sp} values for f''_w in Table 6. An occasional 1 or 2 appears in the fourth decimal place. (The fifth decimal place is rounded.) Finally, we have compared our results with that previously published in Refs. (Back, 1970; Pade et al., 1985; Cohen and Reshotko, 1956; Back, 1976; Narayana and Ramanoorthy, 1972; Wortman and Mills, 1971) When overlap exists, agreement is excellent. For instance, Table 1 lists the separation values of Cohen and Reshotko (Cohen and Reshotko, 1956) in the last column.

From Eq. (5) we have that δ^* is negative when

$$(1-g_w)\,C_t-C_v>\frac{Sf''_w}{1+(1-S)\beta} \tag{15}$$

As is well known, δ^* is negative when g_w is small and β is large. This result stems from the relatively large density in the boundary layer. However, the inequality also holds when $g_w=0.2$, $\beta=1.25$, and S is 0.038 or less. When $g_w\geq 1$, Eq. (5) shows that δ^* cannot be negative. Similarly, from Eq. (11) θ is negative when

$$C_v-(1-g_w)\,C_t>\frac{f''_w}{\beta} \tag{16}$$

This inequality holds when there is sufficient velocity overshoot, for instance, when $g_w=1.5$ and $\beta=10$. It is easy to see that δ, δ^*, and ϕ are never negative.

The incompressible limit is simply given by

$$\beta=\beta_i,\ g_w=1,\ S=0,\ g(\eta)=1$$

where $f(\eta)$ is determined by Eq. (2a), which is the Falkner-Skan equation when $g_w=1$.

When the wall is adiabatic, G is undefined, although $g(\eta)=1$ and h_o is a constant across the boundary layer. As a consequence, C_t and G'_w are discarded. However, the results for f''_w, c_f, C_v, and η_{ev} are unaltered, and thus hold for an adiabatic wall. An adiabatic-wall temperature profile

$$\frac{T}{T_e}=\frac{1-Sf'^2}{1-S} \tag{17}$$

is used in conjunction with a temperature thickness, $\tilde{T}_{et}$, defined by,

$$\frac{T_{oe}-\tilde{T}_{et}}{T_{oe}-T_e}=0.9801 \tag{18}$$

where T_{oe} is the temperature of the gas that is adjacent to the wall, and a tilde denotes an adiabatic wall. Equations (17) and (18) result in

$$f'(\tilde{\eta}_{et})=0.99 \tag{19}$$

Hence, $\tilde{\eta}_{et}$ is given by

$$\tilde{\eta}_{et}=\eta_{ev}(\beta,\ 1) \tag{20}$$

We chose the 0.9801 value in Eq. (18) so that

Table 2 $C_v(\beta, g_w)$

β	g_w										
	0.0	0.2	0.4	0.6	0.8	1.0	1.5	2.0	3.0	4.0	5.0
sp	3.4554	2.9267	2.6691	2.5199	2.4246	2.3580	2.2597	2.2051	2.1473	2.1177	2.0989
.5sp	1.3383	1.3814	1.4079	1.4615	1.5217	1.4408	1.4499	1.4541	1.4599	1.4594	1.4601
0.00	1.2168	1.2168	1.2168	1.2168	1.2168	1.2168	1.2168	1.2168	1.2168	1.2168	1.2168
0.25	1.1145	1.0767	1.0411	1.0075	0.9756	0.9453	0.8752	0.8119	0.7008	0.6051	0.5207
0.50	1.0529	0.9947	0.9416	0.8926	0.8473	0.8047	0.7085	0.6243	0.4807	0.3605	0.2566
0.75	1.0107	0.9391	0.8749	0.8165	0.7629	0.7135	0.6029	0.5073	0.3467	0.2138	0.0999
1.00	0.9793	0.8979	0.8261	0.7612	0.7023	0.6479	0.5279	0.4250	0.2535	0.1126	-0.0076
1.25	0.9549	0.8660	0.7883	0.7187	0.6557	0.5979	0.4711	0.3629	0.1837	0.0373	-0.0874
1.50	0.9350	0.8401	0.7577	0.6843	0.6183	0.5580	0.4260	0.3140	0.1290	-0.0216	-0.1496
1.75	0.9185	0.8186	0.7323	0.6561	0.5875	0.5250	0.3891	0.2740	0.0846	-0.0693	-0.1999
2.00	0.9044	0.8002	0.7108	0.6321	0.5615	0.4975	0.3582	0.2406	0.0476	-0.1089	-0.2415
3.00	0.8642	0.7476	0.6493	0.5638	0.4877	0.4190	0.2710	0.1470	-0.0553	-0.2187	-0.3567
4.00	0.8382	0.7133	0.6094	0.5197	0.4403	0.3689	0.2158	0.0881	-0.1196	-0.2868	-0.4279
5.00	0.8196	0.6887	0.5808	0.4881	0.4066	0.3334	0.1769	0.0467	-0.1644	-0.3342	-0.4772
10.00	0.7708	0.6233	0.5050	0.4052	0.3182	0.2408	0.0765	-0.0591	-0.2780	-0.4532	-0.6006
15.00	0.7483	0.5924	0.4694	0.3665	0.2772	0.1980	0.0307	-0.1070	-0.3287	-0.5060	-0.6550
20.00	0.7347	0.5734	0.4475	0.3428	0.2523	0.1721	0.0032	-0.1357	-0.3589	-0.5372	-0.6871
30.00	0.7186	0.5503	0.4211	0.3142	0.2223	0.1411	-0.0297	-0.1698	-0.3946	-0.5740	-0.7248
40.00	0.7090	0.5363	0.4050	0.2970	0.2042	0.1224	-0.0494	-0.1901	-0.4157	-0.5958	-0.7470
50.00	0.7024	0.5266	0.3939	0.2851	0.1918	0.1096	-0.0628	-0.2039	-0.4301	-0.6105	-0.7621
100.00	0.6862	0.5019	0.3659	0.2552	0.1607	0.0777	-0.0961	-0.2380	-0.4654	-0.6466	-0.7987

Table 3 $C_t(\beta, g_w)$

β	g_w										
	0.0	0.2	0.4	0.6	0.8	1.0	1.5	2.0	3.0	4.0	5.0
sp	2.1374	1.9006	1.7930	1.7328	1.6951	1.6690	1.6310	1.6101	1.5881	1.5767	1.5697
.5sp	1.2570	1.2727	1.2828	1.3026	1.3252	1.2962	1.3002	1.3023	1.3050	1.3051	1.3056
0.00	1.2168	1.2168	1.2168	1.2168	1.2168	1.2168	1.2168	1.2168	1.2168	1.2168	1.2168
0.25	1.1829	1.1696	1.1572	1.1456	1.1348	1.1246	1.1014	1.0810	1.0463	1.0175	0.9929
0.50	1.1623	1.1418	1.1235	1.1070	1.0918	1.0779	1.0473	1.0214	0.9791	0.9454	0.9175
0.75	1.1479	1.1227	1.1008	1.0813	1.0637	1.0478	1.0135	1.9849	0.9393	0.9036	0.8745
1.00	1.1370	1.1085	1.0840	1.0625	1.0434	1.0262	0.9895	0.9594	0.9119	0.8753	0.8456
1.25	1.1285	1.0973	1.0708	1.0479	1.0276	1.0095	0.9713	0.9402	0.8916	0.8544	0.8244
1.50	1.1215	1.0881	1.1602	1.0361	1.0150	0.9962	0.9569	0.9250	0.8756	0.8381	0.8079
1.75	1.1156	1.0805	1.0512	1.0263	1.0045	0.9852	0.9450	0.9126	0.8627	0.8249	0.7947
2.00	1.1106	1.0739	1.0436	1.0180	0.9956	0.9759	0.9350	0.9023	0.8519	0.8140	0.7837
3.00	1.0959	1.0547	1.0215	0.9938	0.9701	0.9493	0.9066	0.8729	0.8217	0.7835	0.7533
4.00	1.0861	1.0419	1.0069	0.9780	0.9534	0.9320	0.8884	0.8542	0.8027	0.7645	0.7344
5.00	1.0791	1.0325	0.9962	0.9665	0.9413	0.9195	0.8754	0.8410	0.7893	0.7512	0.7212
10.00	1.0599	1.0071	0.9674	0.9356	0.9092	0.8865	0.8413	0.8065	0.7548	0.7171	0.6876
15.00	1.0508	0.9947	0.9535	0.9209	0.8939	0.8710	0.8254	0.7906	0.7391	0.7017	0.6725
20.00	1.0452	0.9870	0.9448	0.9118	0.8846	0.8615	0.8158	0.7810	0.7297	0.6925	0.6635
30.00	1.0384	0.9775	0.8343	0.9007	0.8732	0.8500	0.8042	0.7695	0.7185	0.6816	0.6528
40.00	1.0344	0.9717	0.9278	0.8940	0.8664	0.8431	0.7973	0.7626	0.7118	0.6751	0.6465
50.00	1.0316	0.9677	0.9234	0.8893	0.8616	0.8383	0.7925	0.7579	0.7072	0.6706	0.6422
100.00	1.0241	0.9569	0.9117	0.8772	0.8494	0.8261	0.7803	0.7460	0.6958	0.6596	0.6315

Table 4 $n_{ev}(\beta, g_w)$

β	g_w										
	0.0	0.2	0.4	0.6	0.8	1.0	1.5	2.0	3.0	4.0	5.0
sp	5.9631	5.4208	5.1427	4.9755	4.8659	4.7879	4.6702	4.6036	4.5321	4.4940	4.4714
.5sp	3.6642	3.7172	3.7465	3.8117	3.8828	3.7770	3.7828	3.7846	3.7834	3.7845	3.7840
0.00	3.4717	3.4717	3.4717	3.4717	3.4717	3.4717	3.4717	3.4717	3.4717	3.4717	3.4717
0.25	3.3039	3.2483	3.1934	3.1388	3.0842	3.0290	2.8872	2.7348	4.6741	3.8609	3.4840
0.50	3.2041	3.1160	3.0280	2.9386	2.8471	2.7501	2.4763	4.2142	3.3704	3.0558	2.8765
0.75	3.1390	3.0287	2.9175	2.8020	2.6784	2.5432	4.3225	3.5213	3.0066	2.7821	2.7663
1.00	3.0930	2.9670	2.8388	2.7023	2.5517	2.3794	3.7633	3.2089	2.8125	2.7115	2.7455
1.25	3.0591	2.9216	2.7801	2.6268	2.4522	2.2448	3.4573	3.0199	2.6841	2.6993	2.7213
1.50	3.0331	2.8867	2.7350	2.5678	2.3718	2.1311	3.2582	2.8882	2.6259	2.6809	2.6982
1.75	3.0125	2.8591	2.6994	2.5207	2.3057	2.0333	3.1149	2.7889	2.6142	2.6630	2.6771
2.00	2.9959	2.8369	2.6707	2.4826	2.2504	1.9479	3.0050	2.7101	2.6017	2.6462	2.6583
3.00	2.9523	2.7791	2.5970	2.3838	2.0993	1.6898	2.7293	2.5035	2.5560	2.5922	2.6002
4.00	2.9270	2.7460	2.5561	2.3302	2.0115	1.5129	2.5723	2.3961	2.5212	2.5544	2.5607
5.00	2.9102	2.7241	2.5298	2.2971	1.9563	1.3819	2.4671	2.3744	2.4949	2.5265	2.5321
10.00	2.8691	2.6706	2.4684	2.2263	1.8494	1.0217	2.2109	2.3094	2.4237	2.4529	2.4573
15.00	2.8508	2.6464	2.4414	2.1978	1.8164	0.8474	2.0996	2.2780	2.3906	2.4191	2.4231
20.00	2.8399	2.6317	2.4251	2.1809	1.7990	0.7398	2.0698	2.2588	2.3706	2.3988	2.4026
30.00	2.8270	2.6139	2.4054	2.1606	1.7792	0.6090	2.0483	2.2362	2.3469	2.3746	2.3781
40.00	2.8193	2.6031	2.3935	2.1483	1.7674	0.5296	2.0354	2.2226	2.3327	2.3600	2.3632
50.00	2.8141	2.5956	2.3853	2.1400	1.7594	0.4748	2.0265	2.2132	2.3229	2.3499	2.3527
100.00	2.7982	2.5744	2.3632	2.1179	1.7388	0.3374	2.0034	2.1884	2.2969	2.3229	2.3244

Table 5 $C_v(\beta, g_w)$

β	g_w										
	0.0	0.2	0.4	0.6	0.8	1.0	1.5	2.0	3.0	4.0	5.0
sp	5.3324	4.8710	4.6533	4.5293	4.4510	4.3965	4.3167	4.2725	4.2259	4.2014	4.1871
.5sp	3.5622	3.5957	3.6167	3.6586	3.7061	3.6436	3.6513	3.6550	3.6600	3.6599	3.6607
0.00	3.4717	3.4717	3.4717	3.4717	3.4717	3.4717	3.4717	3.4717	3.4717	3.4717	3.4717
0.25	3.3959	3.3672	3.3403	3.3151	3.2914	3.2689	3.2174	3.1715	3.0925	3.0257	2.9680
0.50	3.3501	3.3061	3.2663	3.2299	3.1963	3.1653	3.0965	3.0371	2.9384	2.8581	2.7905
0.75	3.3183	3.2643	3.2165	3.1736	3.1345	3.0988	3.0206	2.9544	2.8464	2.7601	2.6883
1.00	3.2947	3.2334	3.1800	3.1325	3.0898	3.0510	2.9670	2.8967	2.7832	2.6935	2.6193
1.25	3.2762	3.2093	3.1516	3.1009	3.0555	3.0145	2.9264	2.8533	2.7362	2.6442	2.5687
1.50	3.2611	3.1897	3.1287	3.0754	3.0280	2.9854	2.8943	2.8191	2.6994	2.6060	2.5294
1.75	3.2485	3.1733	3.1096	3.0543	3.0054	2.9615	2.8680	2.7913	2.6697	2.5751	2.4979
2.00	3.2379	3.1594	3.0935	3.0365	2.9863	2.9413	2.8460	2.7681	2.6450	2.5496	2.4718
3.00	3.2069	3.1192	3.0469	2.9853	2.9316	2.8839	2.7839	2.7029	2.5762	2.4789	2.4001
4.00	3.1867	3.0928	3.0165	2.9522	2.8964	2.8472	2.7445	2.6618	2.5334	2.4352	2.3559
5.00	3.1722	3.0737	2.9947	2.9284	2.8713	2.8210	2.7166	2.6330	2.5035	2.4048	2.3253
10.00	3.1338	3.0227	2.9364	2.8655	2.8052	2.7526	2.6446	2.5590	2.4276	2.3283	2.2487
15.00	3.1159	2.9984	2.9089	2.8360	2.7744	2.7210	2.6116	2.5254	2.3936	2.2943	2.2148
20.00	3.1050	2.9835	2.8920	2.8180	2.7557	2.7018	2.5915	2.5050	2.3732	2.2740	2.1947
30.00	3.0921	2.9653	2.8714	2.7962	2.7331	2.6787	2.5679	2.4812	2.3491	2.2501	2.1710
40.00	3.0844	2.9542	2.8590	2.7830	2.7195	2.6649	2.5537	2.4668	2.3347	2.2358	2.1568
50.00	3.0791	2.9465	2.8504	2.7740	2.7102	2.6554	2.5438	2.4569	2.3248	2.2259	2.1471
100.00	3.0605	2.9220	2.8247	2.7465	2.6828	2.6281	2.5156	2.4292	2.2987	2.2003	2.1218

Table 6 $f''_w(\beta, g_w)$

β	g_w										
	0.0	0.2	0.4	0.6	0.8	1.0	1.5	2.0	3.0	4.0	5.0
sp	0.0001	0.0001	0.0001	0.0001	0.0000	0.0002	0.0002	0.0000	0.0000	0.0002	0.0000
.5sp	0.4063	0.3743	0.3525	0.3178	0.2812	0.3203	0.3089	0.3024	0.2942	0.2918	0.2896
0.00	0.4696	0.4696	0.4696	0.4696	0.4696	0.4696	0.4696	0.4696	0.4696	0.4696	0.4696
0.25	0.5344	0.5757	0.6161	0.6555	0.6941	0.7319	0.8238	0.9121	1.0905	1.2399	1.3924
0.50	0.5811	0.6550	0.7262	0.7952	0.8623	0.9277	1.0849	1.2348	1.5177	1.7836	2.0366
0.75	0.6181	0.7198	0.8173	0.9112	1.0021	1.0904	1.3019	1.5026	1.8799	2.2332	2.5686
1.00	0.6489	0.7755	0.8963	1.0122	1.1241	1.2326	1.4916	1.7367	2.1963	2.6259	3.0332
1.25	0.6754	0.8249	0.9668	1.1027	1.2336	1.3603	1.6622	1.9473	2.4810	2.9792	3.4511
1.50	0.6987	0.8695	1.0310	1.1854	1.3338	1.4772	1.8185	2.1403	2.7420	3.3031	3.8342
1.75	0.7196	0.9104	1.0903	1.2618	1.4266	1.5857	1.9636	2.3196	2.9845	3.6039	4.1900
2.00	0.7386	0.9483	1.1456	1.3334	1.5135	1.6872	2.0996	2.4877	3.2118	3.8859	4.5236
3.00	0.8013	1.0790	1.3382	1.5836	1.8182	2.0439	2.5781	3.0793	4.0121	4.8790	5.6980
4.00	0.8502	1.1874	1.5003	1.7954	2.0769	2.3473	2.9857	3.5836	4.6946	5.7257	6.6992
5.00	0.8907	1.2816	1.6427	1.9824	2.3056	2.6158	3.3469	4.0307	5.2998	6.4765	7.5869
10.00	1.0308	3.6422	2.1980	2.7162	3.2066	3.6752	4.7751	5.7995	7.6948	9.4477	11.0994
15.00	1.1231	2.9114	2.6208	3.2789	3.8996	4.4915	5.8774	7.1656	9.5449	11.7428	13.8125
20.00	1.1935	2.1349	2.9757	3.7528	4.4843	5.1807	6.8089	8.3203	11.1091	13.6833	16.1063
30.00	1.2997	2.5045	3.5688	4.5472	5.4655	633827	8.7442	10.2614	13.7387	16.9457	19.9628
40.00	1.3805	2.8124	4.0670	5.2164	6.2930	7.3148	9.6959	11.9002	15.9592	19.7006	23.2193
50.00	1.4463	3.0815	4.5051	5.8057	7.0221	8.1755	10.8611	13.3453	17.9173	22.1299	26.0911
100.00	1.6701	4.1255	6.2186	8.1158	9.8829	11.5545	15.4370	19.0214	25.6091	31.6733	37.3727

Table 7 $G'_w(\beta, g_w)$

β	g_w										
	0.0	0.2	0.4	0.6	0.8	1.0	1.5	2.0	3.0	4.0	5.0
sp	0.2479	0.2826	0.3013	0.3128	0.3204	0.3259	0.3342	0.3389	0.3441	0.3428	0.3485
.5sp	0.4530	0.4466	0.4425	0.4348	0.4263	0.4371	0.4354	0.4445	0.4334	0.4333	0.4331
0.00	0.4696	0.4696	0.4696	0.4696	0.4696	0.4696	0.4696	0.4696	0.4696	0.4696	0.4696
0.25	0.4846	0.4909	0.4970	0.5027	0.5082	0.5135	0.5258	0.5371	0.5573	0.5750	0.5909
0.50	0.4942	0.5045	0.5140	0.5228	0.5311	0.5390	0.5569	0.5729	0.6007	0.6247	0.6457
0.75	0.5012	0.5143	0.5262	0.5371	0.5473	0.5568	0.5783	0.5972	0.6298	0.6574	0.6816
1.00	0.5067	0.5219	0.5357	0.5482	0.5597	0.5705	0.5945	0.6156	0.6515	0.6817	0.7081
1.25	0.5111	0.5281	0.5433	0.5571	0.5698	0.5815	0.6075	0.6302	0.6687	0.7009	0.7289
1.50	0.5148	0.5334	0.5498	0.5646	0.8781	0.5906	0.6183	0.6423	0.6828	0.7167	0.7460
1.75	0.5179	0.5378	0.5553	0.5710	0.5853	0.5984	0.6275	0.6526	0.6948	0.7300	0.7604
2.00	0.5206	0.5417	0.5601	0.5766	0.5915	0.6052	0.6354	0.6615	0.7052	0.7414	0.7728
3.00	0.5289	0.5536	0.5748	0.5935	0.6104	0.6258	0.6594	0.6882	0.7361	0.7756	0.8096
4.00	0.5346	0.5619	0.5851	0.6054	0.6236	0.6401	0.6760	0.7066	0.7573	0.7990	0.8347
5.00	0.5389	0.5682	0.5929	0.6144	0.6335	0.6509	0.6885	0.7204	0.7731	0.8164	0.8534
10.00	0.5510	0.5866	0.6157	0.6406	0.6625	0.6822	0.7245	0.7600	0.8182	0.8657	0.9062
15.00	0.5572	0.5963	0.6277	0.6543	0.6776	0.6985	0.7431	0.7804	0.8413	0.8908	0.9330
20.00	0.5611	0.6027	0.6356	0.6633	0.6875	0.7091	0.7551	0.7935	0.8561	0.9069	0.9501
30.00	0.5660	0.6108	0.6456	0.6748	0.7001	0.7226	0.7704	0.8101	0.8748	0.9271	0.9716
40.00	0.5691	0.6160	0.6521	0.6821	0.7080	0.7311	0.7801	0.8206	0.8865	0.9399	0.9851
50.00	0.5712	0.6197	0.6567	0.6873	0.7137	0.7372	0.7869	0.8280	0.8948	0.9488	0.9947
100.00	0.5768	0.6297	0.6689	0.7012	0.7288	0.7533	0.8049	0.8475	0.9165	0.9722	1.0294

additional tables for an adiabatic wall are unnecessary.

From Eq. (A-10), we have

$$\gamma\tilde{\delta}_t=\int_0^{\eta_{et}}\frac{1-Sf'^2}{1-S}d\eta \quad (21)$$

which becomes

$$\gamma\tilde{\delta}_t=\eta_{ev}(\beta,\ 1)+\frac{S}{(1+\beta)(1-S)}(f''_w+C_v) \quad (22)$$

Aside from the replacement of $\tilde{\eta}_{et}$ with η_{ev} $(\beta,\ 1)$, this agrees with Eq. (7) with $g_w=1$.

For consistency with Eq. (18), the G factor in ϕ is replaced with

$$\frac{T_{oe}-T}{T_{oe}-T_e}=f'^2 \quad (23)$$

and Eqs. (A-13) becomes

$$\gamma\tilde{\phi}=\int_0^{\infty}f'(1-f'^2)\,d\eta \quad (24)$$

Part of the integral is given by Eq. (B-3), which involves a second integral. However, the f''^2 integral has not been tabulated, and thus only $\tilde{\eta}_{et}$ and $\tilde{\delta}_t$ are appropriate for an adiabatic wall.

6. Conclusions

A comprehensive set of tables are presented which can be used to determine the five common boundary-layer thicknesses, heat transfer, and skin-friction. Tables are presented as function of a pressure gradient parameter β and a wall to inviscid stagnation temperature ratio g_w. Once we know β, g_w and speed parameter S the required quantities are easily determined by selecting the corresponding values from the tables. Although very fast computers are available nowadays and solving the fifth-order ordinary differential equations are not that difficult, it is still cumbersome to resort to computer whenever we need to know the boundary-layer properties. Furthermore the laminar boundary-layer properties can be used as a benchmark value for the bounding value of the full Navier-Stokes equations.

The quasilinearization method was used to linearize the nonlinear ordinary differential equation. It results in significant reductions in computation time.

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Appendix A

As e subscript denotes the boundary-layer edge, a w subscript denotes the wall, and a zero subscript denotes a stagnation quantity. The usual coordinate transformation applies

$$\xi(s)=\int_0^s (\rho\mu u)_e r_w^{2\sigma} ds \qquad \text{(A-1a)}$$

$$\eta(s,n)=\frac{r_w^{\sigma}(\rho\mu)_e}{(2\xi)^{1/2}}\int_0^n \frac{\rho}{\rho_e}dn \qquad \text{(A-1b)}$$

where

s: arc length along the wall

n: coordinate normal to the wall

ρ: density

μ: viscosity

u: velocity component parallel to the wall

σ: 0 for two-dimensional flow and 1 for axisymmetric flow

t: radial coordinate in an axisymmetric flow

The parameters and dependent variables in Eqs. (2) are

$$\frac{u}{u_e}=\frac{df}{d\eta}=f'(\eta),\ \frac{h_0}{h_{oe}}=g(\eta)$$

$$G=\frac{g-g_w}{1-g_w}=\frac{T_o-T_w}{T_{oe}-T_w},\ g_w=\frac{T_w}{T_{oe}}$$

where h and T are the enthalpy and temperature, respectively. The density ratio in Eq. (A-1) is given by

$$\frac{\rho_e}{\rho}=\frac{T}{T_e}$$

$$=\left(1+\frac{\gamma-1}{2}M_e^2\right)g-\frac{\gamma-1}{2}M_e^2 f'^2 \qquad \text{(A-2)}$$

where γ is the constant ratio of specific heats and M is the Mach number.

The dependence on both M_e and γ is removed by introducing a speed parameter

$$S=\frac{(\gamma-1)M_e^2/2}{1+(\gamma-1)M_e^2/2} \qquad \text{(A-3)}$$

which transforms Eq. (A-2) into

$$\frac{\rho_e}{\rho}=\frac{g-Sf'^2}{1-S}=\frac{g_w+(1-g_w)G-Sf'^2}{1-S} \qquad \text{(A-4)}$$

The pressure gradient parameter can be written as

$$\beta=\beta_i\left(1+\frac{\gamma-1}{2}M_e^2\right)=\frac{\beta_i}{1-S} \qquad \text{(A-5a)}$$

where the incompressible definition of β is

$$\beta_i=\frac{2\xi}{u_e}\frac{du_e}{d\xi} \qquad \text{(A-5b)}$$

In addition to S and ξ, a scaled wall length

$$\bar{x}=\frac{\xi}{(\rho\mu u)_e r_w^{2\sigma}}$$

is used in the Reynolds number

$$Re_{\bar{x}}=\left(\frac{\rho u}{\mu}\right)_e \bar{x}$$

It is convenient to introduce the parameter

$$\gamma=\left(\frac{Re_x}{2}\right)\frac{1}{\bar{x}}=\frac{(\rho u)_e r_w{}^{\sigma}}{(2\xi)^{1/2}} \qquad \text{(A-6)}$$

Equation (A-1b) can be inverted to yield

$$\gamma n=\int_0^{\eta}\frac{\rho_e}{\rho}d\eta=\int_0^{\eta}\frac{g_w+(1-g_w)-Sf'^2}{1-S}d\eta$$

where Eq. (A-4) is used for the density ratio.

The heat transfer q_w and Stanton number St are given by

$$q_w=k_w\left(\frac{\partial T}{\partial n}\right)_w=\frac{h_{oe}-h_{ow}}{(2Re_{\bar{x}})^{1/2}}(\rho u)_e G'_w$$

$$St=\frac{q_w}{(h_{oe}-h_{ow})(\rho u)_e}=\frac{G'_w}{(2Re_{\bar{x}})^{1/2}} \qquad \text{(A-7)}$$

where k is the thermal conductivity of the gas. The skin-friction and skin-friction coefficient are

$$\tau_w=\mu_w\left(\frac{\partial u}{\partial n}\right)_w=\frac{(\rho u^2)_e}{(2Re_{\bar{x}})^{1/2}}f''_w \qquad \text{(A-8a)}$$

$$c_f = \frac{2\tau_w}{(\rho u^2)_e} = \left(\frac{2}{Re_{\bar{x}}}\right)^{1/2} f''_w \qquad \text{(A-8b)}$$

From Eqs. (A-7) and (A-8b), we obtain the Reynolds-analogy relation

Five boundary-layer thicknesses are defined as follows:

δ = velocity thickness = n when $f' = 0.99$ and $\eta = \eta_{ev}(\beta, g_w)$

$$\gamma\delta = \int_0^{\eta_{ev}} \left[\frac{g_w + (1-g_w) - Sf'^2}{1-S}\right] d\eta \qquad \text{(A-9)}$$

δ_t = thermal thickness = n when G = 0.99 and $\eta = \eta_{et}(\beta, g_w)$

$$\gamma\delta_t = \int_0^{\eta_{et}} \left[\frac{g_w + (1-g_w) - Sf'^2}{1-S}\right] d\eta \qquad \text{(A-10)}$$

δ^* = displacement thickness

$$= \int_0^\infty \left[1 - \frac{\rho u}{(\rho u)_e}\right] dn$$

$$\gamma\delta^* = \int_0^\infty \left[\frac{g_w + (1-g_w) - Sf'^2}{1-S} - f'\right] d\eta \qquad \text{(A-11)}$$

θ = momentum defect thickness

$$= \int_0^\infty \frac{\rho u}{(\rho u)_e}\left(1 - \frac{u}{u_e}\right) dn$$

$$\gamma\theta = \int_0^\infty f'(1-f')\, d\eta \qquad \text{(A-12)}$$

ϕ = stagnation enthalpy defect thickness

$$= \int_0^\infty \frac{\rho u}{(\rho u)_e}\left(1 - \frac{h_o - h_{ow}}{h_{oe} - h_{ow}}\right) dn$$

$$\gamma\phi = \int_0^\infty f'(1-G)\, d\eta \qquad \text{(A-13)}$$

The velocity and thermal transformed edge values, η_{ev} and η_{et} are functions only of β and g_w. (The 0.99 value for f' in the δ definitions is used when there is no overshoot.) On the other hand the right sides of Eqs (A-9)-(A-11) are also functions of S as shown in Sec 3. Of the thicknesses, θ and ϕ are independent of S.

Appendix B

The derivation in Sec. 3 requires the following integrals:

$$\int_0^\eta d\eta = \eta,\ \int_0^\eta f' d\eta = f,\ \int_0^\eta f'' d\eta = f'$$

$$\int_0^\eta f''' d\eta = f'' - f''_w,$$

$$\int_0^\eta ff'' d\eta = ff' - \int_0^\eta f'^2 d\eta$$

$$\int_0^\eta Gf d\eta = fG + G' - G'_w \qquad \text{(B-1)}$$

$$\int_0^\eta f'^2 d\eta = \frac{1}{1+\beta}(f'' + ff' + \beta g_w \eta - f''_w) + \frac{\beta(1-g_w)}{1+\beta}\int_0^\eta G d\eta \qquad \text{(B-2)}$$

$$\int_0^\eta f'^3 d\eta = \frac{2}{1+2\beta}\Big[f'f'' + \frac{1}{2}ff'^2 + \beta f + \beta(1-g_w)(fG + G' - G'_w) - \int_0^\eta f''^2 d\eta\Big] \qquad \text{(B-3)}$$

These relations utilize Eqs. (2c) and (2d). The first four equations are self-evident, while the last four involve an integration by parts. For example, to obtain Eq. (B-3), multiply Eq. (2a) by $f'\ d\eta$ and integrate from zero to η, with the result

$$\int_0^\eta f'f''' d\eta + \int_0^n ff'f'' d\eta + \beta g_w \int_0^\eta f' d\eta + \beta(1-g_w)\int_0^\eta Gf' d\eta - \beta\int_0^\eta f'^3 d\eta = 0 \qquad \text{(B-4)}$$

The more difficult integrals are evaluated as follows:

$$\int_0^\eta f'f''' d\eta = \int_0^n f' df'' = f'f'' - \int_0^\eta f''^2 d\eta \qquad \text{(B-5)}$$

$$\int_0^\eta ff'f'' d\eta = \int_0^\eta ff' df' = \frac{1}{2}f'^2 - \frac{1}{2}\int_0^\eta f'^3 d\eta \qquad \text{(B-6)}$$

$$\int_0^\eta Gf' d\eta = \int_0^\eta G df = fG + \int_0^\eta G'' d\eta = fG + G' - G'_w \qquad \text{(B-7)}$$

In the last integral, Eq. (2b) is used and Eq. (B-1) is verified. Combining the above relations with Eq. (B-4) yields Eq. (B-3).